

# An algebraic $SU(1, 1)$ solution for the relativistic hydrogen atom

R. P. Martínez-y-Romero

*Facultad de Ciencias, Universidad Nacional Autónoma de México, Apartado Postal 21-267, C P 04000, Coyoacán D. F. México.*

H. N. Núñez-Yépez

*Departamento Física, Universidad Autónoma Metropolitana, Unidad Iztapalapa, Apartado Postal 55-534, C P 09340, Iztapalapa D. F. México.*

A. L. Salas-Brito

*Laboratorio de Sistemas Dinámicos, Departamento de Ciencias Básicas, Universidad Autónoma Metropolitana, Unidad Azcapotzalco, Apartado Postal 21-267, C P 04000, Coyoacán D. F. México.*

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## Abstract

The bound eigenfunctions and spectrum of a Dirac hydrogen atom are found taking advantage of the  $SU(1, 1)$  Lie algebra in which the radial part of the problem can be expressed. For defining the algebra we need to add to the description an additional angular variable playing essentially the role of a phase. The operators spanning the algebra are used for defining ladder operators for the radial eigenfunctions of the relativistic hydrogen atom and for evaluating its energy spectrum. The status of the Johnson-Lippman operator in this algebra is also investigated.

*Key words:* relativistic hydrogen atom, ladder operators,  $SU(1, 1)$  Lie algebra  
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The bound solutions of the hydrogen atom are of great importance in both classical and quantum mechanics and so is the search for new ways of solving or using such problem [1,2,3,4,5,6,7,8,9,10]. The purpose of this paper is to discuss an algebraic solution for the bounded eigenstates of the relativistic hydrogen

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*Email addresses:* `rmr@hp.fciencias.unam.mx` (R. P. Martínez-y-Romero), `nyhn@xanum.uam.mx` (H. N. Núñez-Yépez), `asb@correo.azc.uam.mx` (A. L. Salas-Brito).

atom based on the  $SU(1,1)$  properties of the extended radial part of the problem [11,13]. The algebra is spanned by two operators  $\Xi_i$ ,  $i = 1, 2$  in terms of which such part of the problem can be expressed, plus an additional—hence, *extended*—phase shifting operator,  $\Xi_3$ , whose only effect is to change the relative phases of the eigenfunctions of the algebra participating in its radial eigenfunctions; though, for maintaining the eigenfunctions of the hydrogen atom, it is found necessary to project our results to vanishing additional phase after applying the operator  $\Xi_3$ . That notwithstanding, the set of operators allow a very direct solution of the relativistic hydrogen problem. Moreover, the solution exhibits that the extended radial system possesses a  $SU(1,1)$  symmetry<sup>1</sup>—the original problem possessing an  $SO(2,1)$  one [14]. We further exhibit how the eigenfunctions of the algebra, *i. e.* the radial eigenfunctions of the relativistic hydrogen atom, can be related to the Laguerre polynomials of noninteger indices and to the Sonine polynomials [15]. These results can be of importance in the study of squeezed and coherent states of the relativistic hydrogen atom, as well as in diverse quantum optics applications [6,16,17,18].

That the relativistic Coulomb problem could be solved exactly and that the energies so calculated showed an excellent agreement with the experiments was indeed one of the reasons for the rapid acceptance of Dirac relativistic quantum mechanics. Almost from the start operational methods were applied to the hydrogen atom, as in [19,20,21]. In due course other methods were developed, for example in [22,23], and in Biedenharn's solution [24] in which advantage was taken of the constant operator  $B$  introduced by Johnson and Lippmann [24,25,26]—shown in equation (42) below.

The Hamiltonian of the relativistic hydrogen atom is

$$H_D = c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta c^2 - \frac{1}{r}, \quad (1)$$

using atomic units ( $\hbar = m = e = 1$ ),  $c$  is the speed of light,  $\boldsymbol{\alpha}$  and  $\beta$  are the Dirac matrices in the Dirac representation [27,28],

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2)$$

where the 1's and 0's stand, respectively, for  $2 \times 2$  unit and zero matrices and the  $\sigma$  is the operator composed by the three Pauli matrices  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ . The method to be discussed in this paper is different from all the aforementioned, as it deals exclusively with the radial part of the problem. Besides, to obtain a system of equations invariant under  $SU(1,1)$ , we need to extend the

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<sup>1</sup> symmetry that can, by suitably changing the definition of the  $\Xi_i$ —hence the commutation relations—be recasted as a  $SU(2)$  one [7,11,12]

configuration space by adding an extra angular variable  $\varphi$  (Cf. equation (17) below).

Since the Hamiltonian (1) is invariant under rotations, we look for simultaneous eigenfunctions of  $H_D$ ,  $|\mathbf{J}|^2$  and  $J_z$ , where  $\mathbf{J} = \mathbf{L} + \mathbf{s}$  is the total angular momentum, and  $\mathbf{s} \equiv \frac{1}{2}\mathbf{\Sigma}$  is the spin. Parity is a good quantum number in our problem, but instead of parity, we use the parity related quantum number  $\epsilon$ , defined as

$$\epsilon = \begin{cases} 1 & \text{If } l = j + \frac{1}{2}, \\ -1 & \text{If } l = j - \frac{1}{2}, \end{cases} \quad (3)$$

we can easily also show that the eigenvalues of the operator  $K \equiv \beta(1 + \mathbf{\Sigma} \cdot \mathbf{L})$ , are given by  $-\epsilon(j + 1/2)$ , so, in a way,  $\epsilon$  is a more useful quantity than parity.

The lower components of the wave function should have an opposite parity than the upper ones, and as parity goes as  $(-1)^l$ , where  $l$  is the orbital angular momentum quantum number, if we define  $l' = j - \epsilon/2$ , we can cast the eigensolutions of (1) as

$$\Psi(r, \theta, \phi) = \frac{1}{r} \begin{pmatrix} F(r) \mathcal{Y}_{j=l-\epsilon/2}^m(\theta, \phi) \\ iG(r) \mathcal{Y}_{j=l'+\epsilon/2}^m(\theta, \phi) \end{pmatrix}. \quad (4)$$

where  $\mathcal{Y}_{j=l\pm\epsilon/2}^m(\theta, \phi)$  are spinorial harmonics [29]. The form proposed for the eigenfunctions solves the angular part of the hydrogen atom.

Let us define the quantity  $k \equiv \sqrt{c^4 - E^2}$ , which is positive definite since we are interested in bound states. Furthermore, let us also define

$$\zeta \equiv \alpha_F = 1/c, \quad \tau_j \equiv \epsilon \left( j + \frac{1}{2} \right), \quad \nu \equiv \sqrt{\frac{c^2 - E}{c^2 + E}}, \quad (5)$$

where  $\alpha_F \simeq 1/137$  is the fine structure constant. With the proposed solution (4), we have to deal only with the radial equation for the relativistic hydrogen atom [30]

$$H_{Dr} = \alpha_r \left[ p_r - i \frac{\tau_j}{r} \beta \right] + c^2 \beta - \frac{1}{r}, \quad (6)$$

where

$$\alpha_r \equiv \frac{1}{r} \boldsymbol{\alpha} \cdot \mathbf{r}, \quad p_r \equiv -\frac{i}{r} \left( 1 + r \frac{d}{dr} \right). \quad (7)$$

Using the radial part of the solution (4), definitions (5) and (7), equation (6), and using  $\rho \equiv kr$ , we can obtain the pair of equations [8]

$$\left(-\frac{d}{d\rho} + \frac{\tau_j}{\rho}\right) G(\rho) = \left(-\nu + \frac{\zeta}{\rho}\right) F(\rho), \quad (8)$$

and

$$\left(+\frac{d}{d\rho} + \frac{\tau_j}{\rho}\right) F(\rho) = \left(\nu^{-1} + \frac{\zeta}{\rho}\right) G(\rho). \quad (9)$$

The above equations, (8) and (9), can be rewritten using a set of three operators whose commutation relations define a  $SU(1, 1)$  algebra, as we exhibit in what follows. Introducing the variable  $x$  through the relation  $\rho = e^x$ , so  $x \in (-\infty, \infty)$ , and the functions  $\psi_{\pm}(x)$  through

$$F(\rho(x)) = \sqrt{c^2 + E} [\psi_-(x) + \psi_+(x)], \quad (10)$$

$$G(\rho(x)) = \sqrt{c^2 - E} [\psi_-(x) - \psi_+(x)], \quad (11)$$

we obtain

$$\left[\frac{d}{dx} + e^x - \frac{\zeta E}{\sqrt{c^4 - E^2}}\right] \psi_+(x) = \left(\frac{\zeta}{\sqrt{c^4 - E^2}} - \tau_j\right) \psi_-(x), \quad (12)$$

and

$$-\left[\frac{d}{dx} - e^x + \frac{\zeta E}{\sqrt{c^4 - E^2}}\right] \psi_-(x) = \left(\frac{\zeta}{\sqrt{c^4 - E^2}} + \tau_j\right) \psi_+(x). \quad (13)$$

This first-order system can be uncoupled to yield

$$\left[\frac{d^2}{dx^2} + 2\mu e^x - e^{2x} - \frac{1}{4}\right] \psi_+(x) = \left(\tau_j^2 - \zeta^2 - \frac{1}{4}\right) \psi_+(x), \quad (14)$$

and

$$\left[\frac{d^2}{dx^2} + 2(\mu - 1)e^x - e^{2x} - \frac{1}{4}\right] \psi_-(x) = \left(\tau_j^2 - \zeta^2 - \frac{1}{4}\right) \psi_-(x), \quad (15)$$

where we have defined

$$\mu \equiv \frac{\zeta E}{\sqrt{c^4 - E^2}} + 1/2. \quad (16)$$

The seemingly odd  $-1/4$  term in (14) and (15), has to be included in order to close the algebra that solves the problem. We pinpoint that the change of variable from  $\rho$  to  $x$  is, strictly speaking, not necessary for any of the calculations that follow; the change, however, simplifies the appearance of some of the equations. Notice also that we can regard the equations (14) and (15) as eigenvalue equations with a common and predetermined eigenvalue  $\xi = \tau_j^2 - \zeta^2 - \frac{1}{4} = j(j+1) - \zeta^2$ . Since the minimum value of  $j$  is  $1/2$ , then  $\xi > 0$ . In cases where the atomic number,  $Z$ , is different from 1—the value we are assuming in this article— then  $\xi \geq 0$  for  $Z = 2, \dots$  up to 118. If  $Z$  exceeds this value then  $\xi$  could become imaginary and the solutions would then exhibit irregular behaviour [26].

To construct the Lie algebra for our system, let us introduce an extra variable,  $\varphi \in [0, 2\pi]$ , through the operator

$$\Xi_3 \equiv -i \frac{\partial}{\partial \varphi}, \quad (17)$$

and then define

$$\Xi_{\pm} \equiv i e^{\pm i \varphi} \left( \frac{\partial}{\partial x} \mp e^x \mp i \frac{\partial}{\partial \varphi} + \frac{1}{2} \right), \quad (18)$$

which depend both on  $\varphi$  and on the transformed ‘radial’ variable  $x$ . These operators satisfy the algebra

$$[\Xi_3, \Xi_{\pm}] = \pm \Xi_{\pm}, \quad [\Xi_+, \Xi_-] = -2\Xi_3. \quad (19)$$

Let us now introduce the operators  $\Xi_1$  and  $\Xi_2$

$$\Xi_1 = \frac{1}{2} (\Xi_+ + \Xi_-), \quad \Xi_2 = \frac{1}{2i} (\Xi_+ - \Xi_-), \quad (20)$$

which together with  $\Xi_3$  satisfy the  $SU(1,1)$  Lie algebra [4,13],

$$[\Xi_1, \Xi_2] = -i\Xi_3, \quad [\Xi_2, \Xi_3] = i\Xi_1, \quad [\Xi_3, \Xi_1] = i\Xi_2; \quad (21)$$

whose Casimir operator is [13]

$$\Xi_c = -\Xi_1^2 - \Xi_2^2 + \Xi_3^2 = \partial^2 / \partial x^2 - e^{2x} - 2ie^x \partial / \partial \varphi - 1/4. \quad (22)$$

Note that we do not have a linear term in  $\partial / \partial x$  in the Casimir operator precisely because we included the  $1/2$ -term in the definition of the  $\Xi_{\pm}$ . It is

also worth noting that the realization of the  $SU(1, 1)$  algebra given in the previous equations is in terms of two variables (another example is described in [31]) and not in terms of a single one as it is more usual [32,33].

To find the simultaneous eigenfunctions and eigenvalues of the operators  $\Xi_c$  and  $\Xi_3$ , let us change to the variable  $x$ , using  $\rho = e^x$ , and write the eigenfunctions as  $V_\xi^\mu(x, \varphi)$ ; where  $\mu$  and  $\xi$  stand for the eigenvalues of  $\Xi_3$  and  $\Xi_c$ , respectively,

$$\begin{aligned}\Xi_3 V_\xi^\mu(x, \varphi) &= \mu V_\xi^\mu(x, \varphi) \\ \Xi_c V_\xi^\mu(x, \varphi) &= \xi V_\xi^\mu(x, \varphi).\end{aligned}\tag{23}$$

Using equation (17), we find

$$V_\xi^\mu(x, \varphi) = e^{i\mu\varphi} P_\xi^\mu(x).\tag{24}$$

It is clear that  $\varphi$  plays the role of a phase, and

$$\psi_+(x) = P_\xi^\mu(x), \quad \psi_-(x) = P_\xi^{\mu-1}(x).\tag{25}$$

From equation (19), we can also conclude that the  $\Xi_\pm$  are ladder operators for the problem, they change  $\mu$  to  $\mu \pm 1$ , so they are easily shown to satisfy

$$\Xi_\pm V_\xi^\mu(x, \varphi) \propto V_\xi^{\mu\pm 1}(x, \varphi).\tag{26}$$

To establish the properties of the  $\Xi$ -operators we need to define the inner product

$$\langle \phi, \psi \rangle = \int_0^{2\pi} \frac{d\varphi}{2\pi} \int_{-\infty}^{\infty} \phi^*(\varphi, x) \psi(\varphi, x) dx,\tag{27}$$

where we are assuming that  $\phi(\varphi, x)$  and  $\psi(\varphi, x)$  are periodic functions over the interval  $\varphi \in [0, 2\pi]$  and that they vanish as  $x \rightarrow \pm\infty$ —so as to make them square-integrable. Using (27), we can prove that  $\Xi_1, \Xi_2$ , and  $\Xi_3$  are hermitian,  $\Xi_i^\dagger = \Xi_i$ ,  $i = 1, 2, 3$ .

To see that the problem of finding the radial eigenfunctions of the extended radial part of the relativistic hydrogen atom is equivalent to finding a representation of the  $SU(1, 1)$  algebra, we introduce a complete orthogonal basis of simultaneous eigenfunctions for  $\Xi_c$  and  $\Xi_3$ , namely  $V_\xi^\mu(x, \varphi) \equiv |\xi \mu \rangle$ , we require that  $\langle \xi' \mu' | \xi \mu \rangle = \delta_{\xi, \xi'} \delta_{\mu, \mu'}$ , using the inner product (27).

The operator  $\Xi^2 = \Xi_1^2 + \Xi_2^2 + \Xi_3^2$  is obviously definite positive; furthermore, as

$$\Xi^2 = 2\Xi_3^2 - \Xi_c, \quad (28)$$

we must have  $2\mu^2 \geq \xi$ , this in turn means that  $|\mu|$  is bounded by below, as a consequence there is a minimum value for it, say  $|\mu|_{min}$ . We have then two choices, either we choose  $\mu > 0$  or we choose  $\mu < 0$ . In the first case  $\mu$  itself is bounded by below; in the second,  $\mu$  is bounded by above.

Let us take first  $\mu > 0$  and define  $\lambda \equiv \mu_{min}$ . Let us call  $\lambda \equiv |\mu|_{min}$  then  $\Xi_-|\xi \lambda \rangle = 0$ , or equivalently  $\Xi_+\Xi_-|\xi \lambda \rangle = 0$ . Since  $\Xi_+\Xi_- = -\Xi_c + \Xi_3^2 - \Xi_3$ , this means

$$-\xi + \lambda^2 - \lambda = 0 \quad \text{or} \quad \xi = \lambda(\lambda - 1). \quad (29)$$

As  $\lambda$  has to be positive, we directly obtain that  $\mu_{min} \equiv \lambda = s + \frac{1}{2}$ , where we have defined  $s \equiv +\sqrt{\tau_j^2 - \zeta^2} = +\sqrt{(j + 1/2)^2 - \alpha_F^2}$ , so  $s$  is real.

Some features of this algebraic approach are worth noting. First, notice that the eigenvalue  $\xi = \lambda(\lambda - 1)$  is not determined by the algebra itself, but just by the angular symmetry of the problem. Second, that this eigenvalue is not necessarily an integer or half-integer number as it should be in the related  $SU(2)$  case. Nevertheless, it is interesting to point out that in the non-relativistic limit  $\lambda$  becomes approximately half-integer since in such a limit we can neglect the fine structure constant  $\alpha_F$ , so  $s \approx (j + 1/2)$ . In this case the eigenvalues of  $\Xi_c$  become  $l(l + 1)$  but the structure of the algebra  $SU(1, 1)$  remains unchanged. The remarkable point here is that the representation of the algebra and the energy spectrum become precisely those of the non-relativistic hydrogen atom [45].

With the value of  $\lambda$  fixed, we proceed to obtain the representations of  $SU(1, 1)$ . We found it convenient to relabel the kets  $|\xi \mu \rangle$ , using  $\xi = \lambda(\lambda - 1)$ , as  $|\lambda \mu \rangle$ . So, starting with  $|\lambda \lambda \rangle$ , we obtain an infinite —since  $\mu$  is bounded only by below— series of states  $|\lambda \lambda + 1 \rangle, |\lambda \lambda + 2 \rangle, \dots$ , by applying  $1, 2, \dots$  times  $\Xi_+$  to the ground state  $|\lambda \lambda \rangle$ . Therefore,  $\mu$  itself takes the values  $\mu = \lambda, \lambda + 1, \lambda + 2, \dots = \lambda + n$ , with  $n = 1, 2, \dots$

To find the energy spectrum, we just need to solve, from definition (16),  $E$  in terms of  $\mu$ ; thus the energy spectrum of the hydrogen atom is

$$E = c^2 \left[ 1 + \frac{\zeta^2}{(\mu - 1/2)^2} \right]^{-1/2} = c^2 \left[ 1 + \frac{\alpha_F^2}{(\mu - 1/2)^2} \right]^{-1/2}, \quad (30)$$

where  $\mu = \lambda + n$ , or  $\mu - 1/2 = s + n$ ,  $s = \sqrt{(j + 1/2)^2 - \alpha_F^2}$ ,  $n = 0, 1, 2, \dots$ .

Let us write the above result in a more familiar form. We define the principal quantum number  $N = 1, 2, 3 \dots$  and the auxiliary quantity  $\epsilon_j$  as follows

$$\begin{aligned} N &\equiv j + 1/2 + n, \\ \epsilon_j &\equiv N - s - n = j + 1/2 - s. \end{aligned} \quad (31)$$

From this definition we have that  $\mu - 1/2 = s + n = N - \epsilon_j$ . When we introduce this last result in the energy equation (30), we obtain precisely the Dirac energy spectrum for the Hydrogen atom,[21,27,28,30,35].

To obtain the eigenfunctions from the representation, let us write

$$\Xi_{\pm} |\lambda \mu\rangle = C_{\mu}^{\pm} |\lambda \mu \pm 1\rangle. \quad (32)$$

We can evaluate these constants from  $\langle \lambda \mu | \Xi_+ \Xi_- | \lambda \mu \rangle = -\lambda(\lambda - 1) + \mu(\mu - 1) = C_{\mu-1}^+ C_{\mu}^-$  and from  $(C_{\mu-1}^+)^* = C_{\mu}^-$ ; if we further assume these constants to be real, we get  $C_{\mu}^{\pm} = \pm \sqrt{\mu(\mu \pm 1) - \lambda(\lambda - 1)}$ .

The ground state wave function is obtained from  $\Xi_- |\lambda \lambda\rangle = 0$ , that is

$$P_{\lambda}^{\lambda}(x) = N_{\lambda} e^{sx} \exp(-e^x) = N_{\lambda} e^{(\lambda-1/2)x} \exp(-e^x) = N_{\lambda} \rho^s e^{-\rho}, \quad (33)$$

where  $N_{\lambda} \equiv \frac{2^{(\lambda-1/2)}}{\sqrt{\Gamma(2\lambda-1)}}$  is the normalization constant calculated using the inner product (27) and  $\Gamma(2\lambda - 1)$  stands for the Euler-gamma function. From equation (25), we find that for the lowest eigenvalue,  $\psi_+(x) = P_{\lambda}^{\lambda}(x)$ , and  $\psi_-(x) = 0$ . In terms of the radial variable  $\rho$ , the ground state solutions are [equation (10)]

$$F(\rho) = N \sqrt{\frac{(c^2 + E)}{2c^2(\lambda - 1/2)}} \rho^s e^{-\rho}, \quad G(\rho) = -N \sqrt{\frac{(c^2 - E)}{2c^2(\lambda - 1/2)}} \rho^s e^{-\rho}. \quad (34)$$

The normalization constant for this state,  $N = 1/\sqrt{2c^2(\lambda - 1/2)}$ , different from  $N_{\lambda}$ , is obtained from the condition

$$\int_0^{\infty} (|F(\rho)|^2 + |G(\rho)|^2) d\rho = 1. \quad (35)$$

The rest of the eigenfunctions for the positive set of eigenvalues  $\mu$  are obtained by applying successively  $\Xi_+$  to the ground state  $|\lambda \lambda\rangle$ . Notice that the value of  $\lambda$  is determined by the total angular momentum and is different for each value of  $j$ .



The representation presented here is called the two-mode bosonic representation, because the operators (18) can be constructed using two bosonic creation operators [46]. This can be easily seen if we define the two mode bosonic operator  $\hat{\lambda} \equiv 1/2(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} + 1)$ , where  $\hat{a}^\dagger$  and  $\hat{b}^\dagger$  are the creation operators of (some) bosonic modes and  $\hat{a}$  and  $\hat{b}$  are the annihilation ones. If we put  $|m, n\rangle$  as the simultaneous eigenstates of the operators  $\hat{a}^\dagger \hat{a}$  and  $\hat{b}^\dagger \hat{b}$ , respectively, we can see that our representation is given by  $|\lambda \mu\rangle = |n + 2\lambda - 1, n\rangle$ , with  $\mu = \lambda + n$ ,  $n = 0, 1, 2, \dots$ . In the non relativistic case  $\lambda$  becomes an integer (the principal quantum number becomes an integer), because  $\lambda = l + 1$ , [45] and the representation becomes what Bargmann calls *minimal M*, ( $\lambda$  has a minimal value),  $D_\lambda^+$ , where  $\lambda$  takes the values  $1/2, 1, 3/2, \dots$  [32]. In the non-relativistic case the semi-integer representations are excluded [45]. It is interesting that in the relativistic case we use a generalization to a non-integer, nor semi-integer index, of the minimal representation.

What happens when we consider the negative set of eigenvalues of  $\Xi_3$  ( $\mu < 0$ )? In that case  $|\mu|_{min} = \lambda$  is the largest eigenvalue, instead of the smallest one. Consequently, we obtain the eigenfunctions by successive applications of  $\Xi_-$  to the ket  $|\lambda \lambda\rangle$ . Starting with this state and repeating the procedure we follow in the  $\mu > 0$  case we find that

$$|\lambda \lambda\rangle \sim e^{-i\lambda\xi} e^{sx} e^{e^x} = e^{-i\lambda\xi} \rho^s e^\rho \quad (36)$$

Which diverges as  $\rho \rightarrow \infty$ . This behavior makes the negative eigenvalues solutions not square integrable, and we must have to discard them if we want to describe physically realizable states, keeping only the positive set of eigenvalues of  $\mu$ .

As a last point, we note that in [7] it is shown that all the solutions are polynomials with weight factor  $W(\rho) = \rho^{\lambda-1/2} e^{-\rho}$ . This weight factor assures that the behavior of the big and the small components of the spinor are regular both at the origin as well as  $\rho \rightarrow \infty$ .

We have thus constructed an  $SU(1,1)$  algebra for the relativistic hydrogen atom by introducing the Hermitian operators  $\Xi_i$ . The eigenstates of the problem are labelled by numbers  $\mu$ , which are neither integers nor half-integers. Let us remark that we were forced to introduce the new variable  $\varphi$  in order to close the algebra. In terms of the solutions of the Dirac equation,  $\varphi$  just plays the role of an extra phase. The functions,  $\psi_\pm$ , defining the eigensolutions of the hydrogen atom [Eqs. (10) and (11)] are, in strict, obtained by the projection to zero additional phase of the eigenfunctions of the algebra, namely

$$\psi_+(x) = P_\xi^\mu(x) = V_\xi^\mu(x, \varphi)|_{\varphi=0}, \quad \psi_-(x) = P_\xi^{\mu-1}(x) = V_\xi^{\mu-1}(x, \varphi)|_{\varphi=0}. \quad (37)$$

What happens to the radial solutions  $F(\rho)$  and  $G(\rho)$  when we apply the op-

erators  $\Xi_c$  and  $\Xi_3$  to the eigensolutions  $V_\xi^\mu(x, \varphi)$ ? Given that both  $V_\xi^\mu(x, \varphi)$  and  $V_\xi^{\mu-1}(x, \varphi)$  are eigenfunctions of  $\Xi_c$ , the energy spectrum and eigenfunctions remain essentially unchanged, excepting for an irrelevant global constant  $\lambda(\lambda-1)$ . But, when we perform a  $\Xi_3$  rotation things are different; in this case we change  $F(x)$  to

$$F(x) \xrightarrow{\Xi_3} [e^{i(\mu-1)\varphi}\psi_-(x) + e^{i\mu\varphi}\psi_+(x)] = e^{i\mu\varphi}[e^{-i\varphi}\psi_-(x) + \psi_+(x)], \quad (38)$$

since  $F(x) \sim [\psi_-(x) + \psi_+(x)]$ . However, the energy spectrum does not change since both  $V_\lambda^\mu(x, \varphi)$  and  $\Xi_3 V_\lambda^\mu(x, \varphi)$  have the same eigenvalue  $\mu$ . Nevertheless, as we can see from equation (38), though the phase  $e^{i\mu\varphi}$  does not play any observable role, the term  $e^{-i\varphi}$  changes the relative phase between the  $\psi_+(x)$  and the  $\psi_-(x)$  and so it changes the radial function  $F(\rho)$  itself. This means that the rotated eigenfunction would no longer be an eigenstate of the radial hydrogen atom Hamiltonian unless we first project the solution to vanishing phase:  $\varphi \rightarrow 0$ . We could say that there should be a sort of superselection rule between the functions  $\psi_-(x)$  and  $\psi_+(x)$ , this interpretation, apart from providing a way out of that minor problem, would also confirm that the operator  $\Xi_3$  cannot be an observable[36,37]. All these peculiar features of our solution method might have some bearings for the properties of certain other systems [38].

We remark that the eigenfunctions  $\psi_+(x) = P_\lambda^\mu(x)$  and  $\psi_-(x) = P_\lambda^{\mu-1}(x)$  are related to the Laguerre polynomials of non integer index [21],  $L_n^{2s}(2e^x)$  and  $L_{n-1}^{2s}(2e^x)$ , as [30]

$$P_\lambda^\mu(x) = ae^{sx}e^{-e^x}L_n^{2s}(2e^x), \quad P_\lambda^{\mu-1}(x) = be^{sx}e^{-e^x}L_{n-1}^{2s}(2e^x) \quad (39)$$

where  $\mu = n + s + 1/2$ , and the constants  $a$  and  $b$  should satisfy

$$\begin{aligned} a(\tau_j + s - \zeta\nu^{-1} + n) + b(n + 2s) &= 0, \\ b(\tau_j - s + \zeta\nu^{-1} - n) - an &= 0. \end{aligned} \quad (40)$$

In fact, if we solve for  $a$  in the second equation and substitute it back in the first of (40), using also that  $s + n = \mu - 1/2$ , we again obtain precisely the energy spectrum of equation (30). Notice that these non-integer Laguerre polynomials are related to the hypergeometric function  ${}_1F_1(-n, \alpha + 1; x)$  and to the Sonine polynomials  $T_\alpha^{(n)}(\rho)$  through the relations [15]

$$\begin{aligned} L_n^\alpha(\rho) &= \frac{\Gamma(\alpha + n + 1)}{n!\Gamma(\alpha + 1)} {}_1F_1(-n, \alpha + 1; \rho) \\ &= (-1)^n \Gamma(\alpha + n + 1) T_\alpha^{(n)}(\rho). \end{aligned} \quad (41)$$

Let us close the paper with some considerations about the Johnson and Lippmann operator,  $B$ , defined as

$$B = \frac{i}{c^2} K \gamma_5 (H_D - \beta c^2) - \frac{1}{c} \Sigma \cdot \hat{\mathbf{r}}, \quad (42)$$

where  $\gamma_5 = i\alpha_1\alpha_2\alpha_3$ , and  $K = \beta(1 + \Sigma \cdot \mathbf{L})$ . This operator was introduced many years ago to study the properties of the Dirac Coulomb Hamiltonian [25,24]. A direct calculation shows that the Hamiltonian of the relativistic hydrogen atom commutes with the Johnson and Lippmann operator, which can be regarded as the relativistic counterpart of the Laplace-Runge-Lenz vector of classical mechanics [34].

To calculate the action of  $B$  over the state functions (4), we need to use the identities  $(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) \mathcal{Y}_{j=l \pm \epsilon/2}^m(\theta, \phi) = -\mathcal{Y}_{j=l \mp \epsilon/2}^m(\theta, \phi)$ ; after some algebra, we found that

$$B\Psi(r, \theta, \phi) = \begin{pmatrix} \left[ \frac{\tau_j}{r} \left( \frac{dF}{dr} + \frac{\tau_j F}{r} - \frac{\zeta G}{r} \right) + \frac{\zeta F}{r} \right] \mathcal{Y}_{j=l'+\epsilon/2}^m(\theta, \phi) \\ \left[ \frac{-i\tau_j}{r} \left( \frac{-dG}{dr} + \frac{\tau_j G}{r} - \frac{\zeta F}{r} \right) + \frac{\zeta iG}{r} \right] \mathcal{Y}_{j=l+\epsilon/2}^m(\theta, \phi) \end{pmatrix}. \quad (43)$$

Here we see that the effect of this operator over the angular part interchanges the coupling of  $l$  with  $l'$  but the total angular momentum quantum number  $j$  remains unchanged. This point explains one of the degeneracies of the wave functions in the relativistic Coulomb problem, since the energy of a particular state depends on  $N$  and  $j$  and not on the coupling of the orbital angular momentum with spin.

We calculate now the effect of  $B$  on the wave function using the equations of motion (8) and (9), changing to the variable  $\rho = kr$  and writing only the effect over the radial part of the function

$$B \begin{bmatrix} \frac{1}{\rho} \begin{pmatrix} F(\rho) \\ iG(\rho) \end{pmatrix} \end{bmatrix} = \frac{1}{c^2 \rho} \begin{pmatrix} c & -i\tau_j(c^2 + E) \\ i\tau_j(c^2 - E) & c \end{pmatrix} \begin{pmatrix} F(\rho) \\ iG(\rho) \end{pmatrix}. \quad (44)$$

As expected, the Johnson and Lippmann operator on shell is no longer a differential operator. Unfortunately, the operator is not diagonal in terms of  $F$  and  $G$ ; nevertheless, using the functions  $\psi_{\pm}$ , we can write the above result as

$$B \begin{bmatrix} \frac{1}{\rho} \begin{pmatrix} F(\rho) \\ iG(\rho) \end{pmatrix} \end{bmatrix} = \frac{1}{\rho} \begin{pmatrix} \sqrt{c^2 + E} \left[ \left( \zeta - \frac{k\tau_j}{c^2} \right) \psi_+ + \left( \zeta + \frac{k\tau_j}{c^2} \right) \psi_- \right] \\ \sqrt{c^2 - E} \left[ -\left( \zeta - \frac{k\tau_j}{c^2} \right) \psi_+ + \left( \zeta + \frac{k\tau_j}{c^2} \right) \psi_- \right] \end{pmatrix}; \quad (45)$$

again we found a non diagonal result, nevertheless, from equation (25), we

know that at least the ground state  $|\lambda\lambda\rangle$  correspond to a diagonal case, because for that state  $\psi_- = 0$ . Are there any other state in which this feature occurs? In order to obtain a diagonal state, we need that one of the two choices  $(\zeta \pm \frac{k\tau_j}{c^2})$  be null. From  $\mu = s + 1/2 + n$ ,  $n = 1, 2, \dots$ ,  $s = \sqrt{\tau_j^2 - \zeta^2}$  and equations (16) and (30), we found that

$$\left(\zeta \pm \frac{k\tau_j}{c^2}\right) = \zeta \left[1 \pm \frac{\tau_j}{\sqrt{(s+n)^2 + \zeta^2}}\right], \quad (46)$$

which is never null excepting again for the ground state  $|\lambda\lambda\rangle$ . But this is *the ground state of the representation of the  $SU(1,1)$  algebra* and not of the hydrogen atom. In terms of the wave functions of the relativistic Coulomb Hamiltonian, the ground state of the algebra corresponds to the infinite set of states. Namely,  $(N = 1 \text{ and } j = 0)$ ,  $(N = 2 \text{ and } j = 1)$ ,  $(N = 3 \text{ and } j = 2)$  and so on. In other words, using spectroscopy notation  $N_l$ , we find the interesting result that the Johnson and Lippmann operator takes a diagonal form when written in terms of  $\psi_{\pm}$ , only for the states 1s, 2p, 3d ...

In conclusion, we have solved exactly the Dirac relativistic hydrogen atom using an algebraic technique based on the  $SU(1,1)$  Lie algebra in terms of which the radial part of the problem can be expressed. With its help we calculated the explicit form of its ground state wave function. We have also derived radial ladder operators for the system and pointed out the way for obtaining all of its radial eigenfunctions using them. Let us also pinpoint that in the limit  $\zeta \rightarrow 0$  the algebra (19) [or (21)] trivializes,  $\mu \rightarrow 1$ , and the spectrum becomes continuous, as corresponds to a free Dirac particle.

The method discussed in this paper, apart from the algebraic solution of the relativistic hydrogen atom, could be of interest in itself as it requires extending the system before the solution and then projecting the results back to zero phase  $\varphi = 0$ —although this last step is not always necessary. It could be an interesting problem to account for all relativistic problems that may be solved by a similar extension and to find their corresponding extended symmetry algebras, as well as to investigate their generalization to  $D > 3$  spatial dimensions [39]. The algebraic method of solution and the expression of the bound state eigenfunctions in terms of the ladder operators  $\Xi_{\pm}$  presented here may offer some simplifications in relativistic atomic physics calculations [40], can be useful in diverse applications in quantum optics [6,17] and it may, possibly, be applied to the study of ionisation states [41] by using non-discrete representations of  $SU(1,1)$ ; at least below the pair-production threshold, or in the non-relativistic limit [42,43,44], where a description of the hydrogen atom in terms of such an  $SU(1,1)$  algebra is also possible [45].

As we already said before, other application of the method described here

concerns the non-relativistic problem [42,43,44], where a description of the hydrogen atom in terms of such an algebra is also possible. The point here is that the  $SU(1, 1)$  algebra remains exactly the same also in the non-relativistic limit, and only the relationship of  $\mu$  with the energy, the value of  $s$  itself, and the fact that we have to deal with only one radial function instead of two, are the only important points we need to change [45].

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